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Reparametrizing Swung Surfaces over the Reals

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Abstract

Let $\mathbb{K} \subseteq \mathbb{R}$ be a computable subfield of the real numbers (for instance, \mathbb{Q}). We present an algorithm to decide whether a given parametrization of a rational swung surface with coefficients in $\mathbb{K}(\mathbf{i})$, can be reparametrized over a real (i.e. embedded in \mathbb{R}) finite field extension of \mathbb{K} . Swung surfaces include, in particular, surfaces of revolution.

keywords: swung surfaces, revolution surfaces, real and complex surfaces, rational parametrization, ultraquadrics.

1 Introduction

A surface of revolution is a surface globally invariant by rotations around a certain line (the axis of revolution). The intersection of the surface with planes containing the revolution axis yields the so called *profile* curves. Revolution surfaces are well known since ancient times and very common objects in Differential Geometry and in Computer Aided Geometric Design. Still, they pose some interesting and challenging questions. One example is the recent work ([18]) devoted to computing the offset of revolution surfaces, provided the generatrix curve of the surface is implicitly given. Another recent paper deals with a new technique for implicitizing rational surfaces of revolution using μ -bases [20]. A basic question, such as efficiently determining, given the implicit equation of an algebraic surface, whether it is, or not, the equation of a surface of revolution, seems unsolved.

On the other hand, in the Geometric Modeling literature, revolution surfaces are often introduced under the assumption that they are generated by a profile plane curve (see e.g. [1], [7], [8]) subject to rotation around some axis. Since circles are rational curves, if the profile curve is rational, the revolution surface obtained by rotating it around a suitable axis will be rational, too. But the converse is not necessarily true (see Example 2.3).

In this paper we will work with *swung surfaces*, which are a natural extension of surfaces of revolution. More precisely, swung surfaces are produced by swinging around the z -axis a **profile curve** in the yz -plane along a **trajectory curve** in the xy -plane, see section 2 for more details. Assume that the

32 profile curve is a plane rational curve parametrized by $(0, \phi_1(t), \phi_2(t))$ and the
 33 trajectory curve is also given by the parametrization $(\psi_1(s), \psi_2(s), 0)$. Then the
 34 corresponding swung surface is parametrized by

$$\mathcal{P}(s, t) = (\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t)) \quad (\ddagger)$$

35 where we assume that the involved rational functions ϕ_i, ψ_j are defined over
 36 $\mathbb{K}(\mathbf{i})$, where \mathbb{K} is a computable subfield of the reals. In fact, in the sequel, for
 37 the purpose of this paper, the equation (\ddagger) above can be taken as the definition
 38 of **rational swung surface \mathcal{S}** .

39 Notice that when the trajectory curve is a circle, say, $(\psi_1(s) = (s^2 - 1)/(1 +$
 40 $s^2), \psi_2(s) = 2s/(1 + s^2), 0)$, the swung surfaces is the revolution surface obtained
 41 by rotating the profile curve around the z -axis. In particular, rational swung
 42 surfaces include all surfaces of revolution generated by rational profile curves,
 43 as well as many other surfaces, e.g. all quadrics. However we do not know
 44 whether every rational revolution surface is a swung surface, in the sense of
 45 having a parametrization of type (\ddagger) , cf. Example 2.3 below. Swung surfaces
 46 are thoroughly used in geometric aided design specially when the profile and
 47 trajectory curves are Bezier curves, and appear as part of the NURBS packages,
 48 see ([10]).

49 Let us describe the problem we will deal with in this framework. Assume
 50 we take as input a swung surface (\ddagger) where the parametrization is given with
 51 coefficients over $\mathbb{K}(\mathbf{i})$, where \mathbb{K} is a computable subfield of the reals (typically,
 52 the field \mathbb{Q} of rational numbers, or an extension of \mathbb{Q} such as $\mathbb{Q}(\sqrt[n]{\alpha})$, with
 53 $\alpha \in \mathbb{Q}^+$), and where \mathbf{i} is the imaginary unit. That is, we suppose the proposed
 54 parametrization has coefficients of the kind $a + b\mathbf{i}$, with $a, b \in \mathbb{K}$. Yet, the swung
 55 surface might have a simpler parametrization, one involving real coefficients
 56 only. Then, our goal is to determine whether there is a change of parameters
 57 simplifying (in the sense of providing real coefficients) the given parametrization
 58 and, if so, to compute this parameter change. An obvious necessary condition
 59 for that is that the surface has “enough” real points. It turns out that in our
 60 case this is also a sufficient condition (see Theorem 4.2 and Corollary 4.5):
 61 the only requirement for the existence of a real reparametrization is that the
 62 surface should be “real”, in the sense of having a two dimensional piece in \mathbb{R}^3
 63 (see Section 3 for precisions on this concept).

64 Let us point out that it is not known, in general, whether a real surface,
 65 provided with a complex parametrization, has as well a real parametrization.
 66 We refer to the introduction of [17] for details on this problem. Therefore our
 67 result is a further step for settling down this general question. The fact that
 68 it is sufficient, in our context, to be real in order to have a real reparametriza-
 69 tion is due, of course, to the close relation of the swung surfaces with a pair
 70 of curves and to the well known fact that, for curves, reality and complex ra-
 71 tionality imply real parametrizability (see [12]). On the other hand, since the
 72 given parametrization of the swung surface does not univocally determine the
 73 associated pair of curves, but just the involved products $\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s)$,
 74 some weaker conditions on these two curves have to be provided, as described
 75 in the statement of Theorem 4.2, item 1.

76 The algorithmic simplification of the coefficients of a parametrization (and
77 more generally, that of simplifying a parametrization by regarding other fea-
78 tures, such as its degree, etc.) is quite involved and has recently deserved quite
79 a bit of attention. We refer the reader, for a detailed description of this general
80 problem and references, to the Introduction of our recent paper [4]. There we
81 have dealt with the case of parametric ruled surfaces, by using an *ad hoc* analy-
82 sis that can not be easily generalized to include other types of surfaces. Yet, it
83 can be said that the approach for the new case of rational swung surfaces shares
84 with the previous one the need to adapt to the particular context the theory of
85 *ultraquadrics* and *hypercircles* (cf. [3], [12], [11]), specifically created to handle
86 over \mathbb{R} the reparametrizing of a given complex parametrization.

87 We must briefly comment on an alternative approach to solve the proposed
88 simplification problem. In fact, it is easy to observe that, given a parametriza-
89 tion (§) over the complexes, the projection onto the z coordinate provides a
90 rational map. Thus, for every value $z = z_0$ we obtain different (perhaps several)
91 values t_0 of t , such that $z_0 = \phi_2(t_0)$ and, then, the fiber over z_0 is one (or more)
92 rational curve $(\phi_1(t_0)\psi_1(s), \phi_1(t_0)\psi_2(s))$. Therefore, following [16] or [17], we
93 are yield to discuss the existence of a real parametrization for this pencil of
94 curves, by reducing it to the case of conics. Roughly speaking, this approach
95 –if it could be carried out– relies on the theoretically well known birationality
96 from rational curves and conics, while our approach, on the other hand, directly
97 establishes such birational map from the family of curves to the –so called, see
98 A– associated Weil variety.

99 One subtle point when dealing with reparametrizations is whether the input
100 parametrization needs to be proper, that is, invertible. Although this is not a
101 problem for curves, since it is well known (Lüroth’s theorem, see, for instance
102 [13]) that the existence of an improper rational mapping implies –and it is
103 algorithmically easy to find– the existence of a birational parametrization [2],
104 this is not the case, in general, for real surfaces (see Example 2.3). In Section 3,
105 we address this issue, in order to allow improper parametrizations as potential
106 inputs for our simplification goal.

107 Thus, we are able to state our main results on the existence and construction
108 of real reparametrizations in the case of non-proper parametrizations of swung
109 surfaces, by requiring, just, the birationality of the parametrizations for the
110 two curves involved in the description of the surface. Starting from any (non-
111 proper) parametrization of a swung surface, it is easy, computationally speaking,
112 to obtain one of the same surface, but verifying the above requirement (through
113 the algorithmic version of Lüroth’s theorem, see [2]).

114 Section 4 contains the general statement for reparametrization of swung
115 surfaces and its proof, relying on some technical aspects which are detailed in
116 an Appendix. Moreover, we include in this Section a simpler reparametrizing
117 statement in the particular case of classical surfaces of revolution. We conclude
118 the paper (Section 5) with some detailed examples and the precise description
119 and discussion of a pair of algorithms, based on our proposed method, as well as
120 a table with running times for the performance of the implemented algorithms
121 on a collection of surfaces. Computations have been obtained using the well

known mathematical software Maple and Sage.

2 Swung Surfaces

As stated above, we will deal in this paper with the family of parametric or rational swung surfaces, that is, surfaces described parametrically in the form

$$(\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$$

where ϕ_i and ψ_j are rational functions over $\mathbb{K}(\mathbf{i})$, where \mathbb{K} is a computable subfield of the reals.

Thus, the intersection of the resulting surface with the planes $z = k$, i.e. perpendicular to the z axis, produces copies of the of the trajectory (ψ) curve dilated with the y values $\phi_2(t_0)$ of the profile curve as augmentation factor. Notice that we obtain as many curves as points of intersection of the given plane with the profile curve, i.e., as solutions t_0 of the equation $\phi_2(t) = k$. Alternatively, consider the plane $y = \lambda x$ that contains the z -axis and take any s_0 such that this plane intersects the trajectory curve at $u_0 = (\psi_1(s_0), \psi_2(s_0), 0)$. Then, referred to the canonical basis of $y = \lambda x$ given by $u_0/||u_0||$ and $e_3 = (0, 0, 1)$, the intersection of the surface with the plane is the curve $||u_0||\phi_1(t)u_0 + \phi_2(t)e_3$ which is the profile curve distorted horizontally by the scalar $||u_0|| = \sqrt{(\psi_1(s_0))^2 + (\psi_2(s_0))^2}$. Thus, if we imagine the profile curve as being joined to the z -axis with a horizontal elastic arm, the surface can be produced mechanically as the contour obtained by stretching ϕ horizontally with factor $\sqrt{(\psi_1(s))^2 + (\psi_2(s))^2}$ as the yz plane rotates or swings around the z -axis.

Since these surfaces are initially described with, perhaps, complex coefficients, we will consider the geometric object defined by the parametrization in \mathbb{C}^3 and, thus, we will denote the surface as $\mathcal{S}_{\mathbb{C}}$. It is important to remark here that the relation between the complex and real parts of this surface will play an important role in what follows. Yet, we want to discard, for the rest of this paper, the case of parametrizations $(\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$ that do not produce a true surface in \mathbb{C}^3 , i.e. such that the Jacobian of the parametrization has, generically, rank smaller than 2. This excludes precisely the following cases:

- when both ϕ_1, ϕ_2 are constant (since, then, ϕ does not describe a true curve)
- when both ψ_1, ψ_2 are constant (since, then, ψ does not describe a true curve)
- when ϕ_1 is identically zero (since, then, $\mathcal{S}_{\mathbb{C}}$ is just a line, the z -axis).
- when ϕ_2 is constant and ψ_2, ψ_1 are proportional (since then $\mathcal{S}_{\mathbb{C}}$ is just a line $\{c_1x = y, z = c_2\}$ or $\{c_1y = x, z = c_2\}$, with c_1, c_2 some constants)

Leaving apart these degenerate cases, this family of surfaces includes, in particular, surfaces of revolution – when the trajectory curve, is the unit circle–

with rational profile curve, but it extends also to other surfaces that are not of revolution, as all quadrics (after a suitably parametrization) as well as other kinds of surfaces, as shown in the following examples.

Example 2.1. Consider a cone with apex at (x_0, y_0, z_0) and a directrix curve parametrized by $(\phi_1(t), \phi_2(t), \phi_3(t))$, so that the cone is the union of straight lines passing through the apex and a point at the directrix. After a suitable translation we may assume that the apex is the origin of coordinates. Then the cone is parametrized as

$$s(\phi_1(t), \phi_2(t), \phi_3(t))$$

Now, considering as new parameter $T = s\phi_3(t)$, we can reparametrize the cone as

$$\left(T \frac{\phi_1(t)}{\phi_3(t)}, T \frac{\phi_2(t)}{\phi_3(t)}, T\right)$$

yielding a parametrization of the kind $(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$. Cones are, then, swung surfaces and our contribution in this paper applies to these surfaces, too, after performing a translation of the apex to the origin.

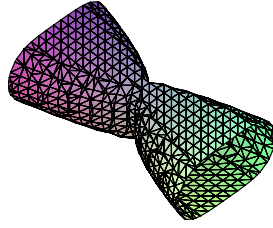


Figure 1: A swung surface: $-y^2 + x^4 + z^2y^2 = 0$

Example 2.2. Let

$$P := (2ts/(t^2 + 1), 2ts^2/(t^2 + 1), (t^2 - 1)/(t^2 + 1))$$

According to our definition this is a parametric swung surface, with profile curve the circle $(0, 2t/(t^2 + 1), (t^2 - 1)/(t^2 + 1))$ and trajectory curve the parabola (s, s^2) . Its implicit equation is $-y^2 + x^4 + z^2y^2 = 0$. See Figure 1.

For another example of this kind, take

$$Q := (2ts/(t^2 + 1), 2ts^2/(t^2 + 1), t^3)$$

182 Again, this is a rational swung surface with profile curve the cubic $(2t/(t^2 + 1), t^3)$, swinging along the parabola (s, s^2) as trajectory curve. See Figure 2.

184 As pointed out previously, surfaces of revolution generated by a rational
 185 curve $(0, \phi_1(t), \phi_2(t))$ are included in the family of parametric swung surfaces.
 186 However, there are surfaces of revolution which are rational, although generated
 187 by non-rational curves, as the following example shows. We do not know yet if
 188 they are parametric swung surfaces.

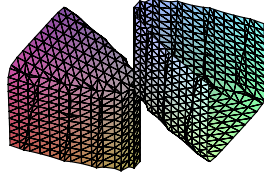


Figure 2: Another swung surface: $x^6 z^2 - 8y^3 z + 6x^4 z y + x^6$

189 **Example 2.3.** Let us consider the offset of an ellipsoid of revolution. See
 190 Figure 3. As explained below, it is known to be rational, but it is also (cf. [18])
 191 the revolution surface generated by the offset curve (which is non-rational) of
 192 an ellipse (cf. [5]). Therefore it is a rational and classical surface of revolution,
 193 which is parametrizable over the reals, yet its intersection with the $x = 0$ plane
 194 (the generatrix curve) is not rationally parametrizable.

195 Indeed, consider the ellipse

$$196 \quad \frac{y^2}{4} + z^2 = 1$$

197 which can be parametrized as

$$198 \quad y = \frac{4t}{t^2 + 1} \quad z = \frac{t^2 - 1}{t^2 + 1}$$

199 We rotate it around the z -axis, so that we get the ellipsoid $\mathcal{S}_{\mathbb{C}}$ (as a surface in
 200 \mathbb{C}^3)

$$201 \quad \frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$$

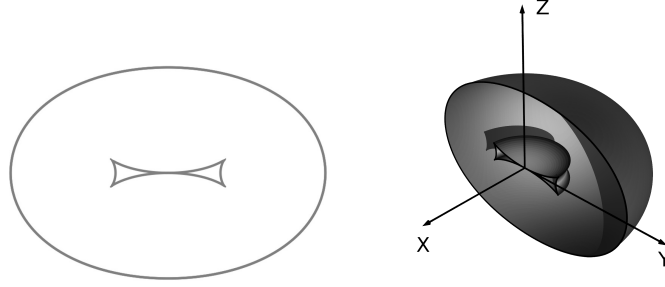


Figure 3: A rational revolution surface, not generated by a rational curve. Left: offset of the ellipse (profile curve of genus 1); Right: half offset of the ellipsoid (rational revolution surface).

which can be parametrized, using the ϕ, ψ scheme, as in the introduction, by

$$x = \frac{4t}{t^2 + 1} \frac{s^2 - 1}{s^2 + 1} \quad y = \frac{4t}{t^2 + 1} \frac{2s}{s^2 + 1} \quad z = \frac{t^2 - 1}{t^2 + 1}$$

or, alternatively (notice that the previous parametrization is not proper), as

$$x = \frac{4t}{t^2 + s^2 + 1} \quad y = \frac{4s}{t^2 + s^2 + 1} \quad z = \frac{t^2 + s^2 - 1}{t^2 + s^2 + 1}$$

For our purposes of constructing the offset of the ellipsoid, an even more suitable parametrization, although one defined over \mathbb{C} , is

$$\begin{aligned} x &= -2 \frac{8t - 16 + s^2 t^2 - 16s^2}{s(t^2 + 8t - 32)} \\ y &= \frac{2i(-8t + 16 + s^2 t^2 - 16s^2)}{s(t^2 + 8t - 32)} \\ z &= \frac{(t - 8)t}{t^2 + 8t - 32}. \end{aligned}$$

Indeed, a mechanical calculation shows that, with this parametrization, the norm of the normal vector to $\mathcal{S}_{\mathbb{C}}$ at a point $(x(t, s), y(t, s), z(t, s))$ is a rational fraction in t and s . Therefore it can be used in a straightforward way to construct the parametric equations of the offset $\mathcal{S}'_{\mathbb{C}}$ at distance 1 of the ellipsoid, which in this way results a rational surface (details on how to compute a rational parametrization of the offset of the ellipsoid can be found in [19], Theorem 5; alternatively, one may check [9]). Namely, we get the following, birational

219 parametrization with complex coefficients of the offset $\mathcal{S}'_{\mathbb{C}}$:

$$\begin{aligned}
220 \quad x &= -1/2 \frac{(5t^2 - 8t + 32)(8t - 16 + s^2t^2 - 16s^2)}{s(t^2 + 8t - 32)(t^2 - 4t + 16)} \\
221 \quad y &= -i/10 \frac{(-8t + 16 + s^2t^2 - 16s^2)(-25t^2 + 40t - 160)}{s(t^2 + 8t - 32)(t^2 - 4t + 16)} \\
222 \quad z &= 2 \frac{t(t+4)(t-2)(t-8)}{(t^2 + 8t - 32)(t^2 - 4t + 16)}. \\
223
\end{aligned}$$

224 Notice that this parametrization is not of the form (\ddagger) of swung surfaces.
225 Also, apparently, the property (known by construction) of $\mathcal{S}'_{\mathbb{C}}$ being real is hid-
226 den behind this birational parametrization. However, implicitization of the
227 above parametrization gives as implicit equation of $\mathcal{S}'_{\mathbb{C}}$:

$$\begin{aligned}
228 \quad &-240y^2z^2x^2 + 66y^2z^4x^2 + 30y^4z^2x^2 + 30y^2z^2x^4 + 450z^2y^2 - 120y^4z^2 - 210y^2z^4 - \\
229 \quad &30y^4x^2 - 30y^2x^4 - 120z^2x^4 - 210z^4x^2 + 450z^2x^2 + 18x^2y^2 + 40y^2z^6 + 10y^6z^2 + \\
230 \quad &33y^4z^4 + 4y^6x^2 + 6y^4x^4 + 4y^2x^6 + 33z^4x^4 + 40z^6x^2 + 10z^2x^6 - 207z^4 - 324z^2 + \\
231 \quad &9x^4 + 9y^4 + 8z^6 - 10y^6 - 10x^6 + 16z^8 + y^8 + x^8 = 0,
\end{aligned}$$

232 which of course is real. Moreover, we know that $\mathcal{S}'_{\mathbb{C}}$ is “real” in the sense that
233 has many real points (see Section 3 for details on this concept), since $(1, 0, 0)$ is
234 a real regular point in this surface.

235 Let us see how we can recover a real parametrization. For that purpose we
236 use the construction of the Weil variety (cf. [3]). In the complex parametriza-
237 tion, we substitute $t = t_0 + it_1$, $s = s_0 + is_1$ and normalize the resulting
238 expressions so that they have real denominators. The Weil variety is then de-
239 fined as the zero set of the imaginary parts of this normal expression, minus
240 the zero set of the denominator (see [3] for further details on this technique for
241 reparametrizing these surfaces over the reals). In our example the Weil variety
242 W is the tubular surface in the hyperplane $t_1 = 0$, described by

$$243 \quad (t_0^2 - 16)s_0^2 + (t_0^2 - 16)s_1^2 - 8(t_0 - 2) = 0,$$

244 and we get an \mathbb{R} -birational map from it to the offset $\mathcal{S}'_{\mathbb{C}}$.

245 Now, by [16], Theorem 3, all tubular surfaces are real parametrizable and,
246 therefore, by composing such parametrization with the mentioned birational
247 map we get a parametrization of $\mathcal{S}'_{\mathbb{C}}$ over the reals. We claim that this real
248 parametrization cannot be birational. Indeed, if it were, by the \mathbb{R} -birational
249 map, our Weil variety W would have a birational parametrization. But following
250 [17], it is easy to deduce that the tubular surface W can not be birationally
251 parametrizable over the reals since its projectivization and desingularization has
252 more than one connected component (an invariant for the real rational function
253 field of the surface, cf. [6]).

254 As a consequence, it follows that the offset $\mathcal{S}'_{\mathbb{C}}$ cannot be birationally parame-
255 trized over the complexes as a swung surface. In fact, were it possible, then, we
256 could apply the Remark 4.3, stating that, under the assumption of $\mathcal{S}'_{\mathbb{C}}$ having
257 a birational complex parametrization as swung surface, the reality of $\mathcal{S}'_{\mathbb{C}}$ would

imply the existence of a birational real parametrization for it, which is not possible as we just pointed out. We remark here that we do not know if there is a complex, non-proper parametrization of $\mathcal{S}'_{\mathbb{C}}$ as swung surface.

On the other hand, we know that, alternatively, $\mathcal{S}'_{\mathbb{C}}$ can be constructed by considering first the offset of the ellipse $(1/4)y^2 + z^2 = 1$ above, which is:

$$-324z^2 + 9y^4 + 450z^2y^2 - 207z^4 - 10y^6 - 120y^4z^2 - 210y^2z^4 + 8z^6 + y^8 + 10y^6z^2 + 33y^4z^4 + 40y^2z^6 + 16z^8 = 0,$$

and then rotating it around the z -axis. However, this curve has genus one (see [5]), so that it is not rational, although its revolution around the z -axis produces the offset $\mathcal{S}'_{\mathbb{C}}$, which, as we have seen, is rational.

In conclusion, $\mathcal{S}'_{\mathbb{C}}$ is a real rational surface of revolution with no rational profile curve for any possible revolution axis, and we do not know whether it can be presented as a parametrized swung surface (although we know that this parametrization can never be proper).

Remark 2.4. More precisely, we have the following: a rational surface of revolution $\mathcal{S}_{\mathbb{C}}$, with the z -axis as the revolution axis, has a sectional curve, at the plane $x = 0$, which is rational if and only if it admits a parametrization

$$(\lambda_1(u, v), \lambda_2(u, v), \lambda_3(u, v))$$

where $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$ is the square of a rational function. Indeed, assume that $\mathcal{S}_{\mathbb{C}}$ is the surface of revolution generated by rotating the planar curve $(0, \phi_1(u), \phi_2(u))$ around the z -axis. Then $\mathcal{S}_{\mathbb{C}}$ has a rational parametrization as

$$\left(\phi_1(u) \frac{v^2 - 1}{1 + v^2}, \phi_1(u) \frac{2v}{1 + v^2}, \phi_2(u) \right)$$

and we have $(\phi_1(u)(v^2 - 1)/(1 + v^2))^2 + (\phi_1(u)2v/(1 + v^2))^2 = \phi_1(u)^2$. Conversely, assume that we have a parametrization

$$(\lambda_1(u, v), \lambda_2(u, v), \lambda_3(u, v))$$

with $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$ the square of a rational function and λ_3 not constant (otherwise $\mathcal{S}_{\mathbb{C}}$ is the plane $z = \lambda_3$). Then, consider a rational curve $(u(t), v(t))$ such that $\lambda_3(u(t), v(t))$ takes, when $t \in \mathbb{C}$, infinitely many values (a property that holds for almost every choice of $(u(t), v(t))$). Now, for almost every t_0 , the point

$$\left(0, \sqrt{\lambda_1(u(t_0), v(t_0))^2 + \lambda_2(u(t_0), v(t_0))^2}, \lambda_3(u(t_0), v(t_0)) \right)$$

lies in $\mathcal{S}_{\mathbb{C}}$, since so does the point

$$(\lambda_1(u(t_0), v(t_0)), \lambda_2(u(t_0), v(t_0)), \lambda_3(u(t_0), v(t_0)))$$

and $\mathcal{S}_{\mathbb{C}}$ contains every circle in a xy -parallel plane with center at $(0, 0, \lambda_3(u(t_0), v(t_0)))$ and passing through $(\lambda_1(u(t_0), v(t_0)), \lambda_2(u(t_0), v(t_0)), \lambda_3(u(t_0), v(t_0)))$.

Now, it is immediate to conclude that the intersection of $\mathcal{S}_{\mathbb{C}}$ with the plane $x = 0$ can be parametrized by

$$\left(0, \sqrt{\lambda_1(u(t), v(t))^2 + \lambda_2(u(t), v(t))^2}, \lambda_3(u(t), v(t))\right)$$

which is rational by our hypothesis on $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$.

In particular it follows that the offset $\mathcal{S}'_{\mathbb{C}}$ of the previous example, can not have a parametrization $(\lambda_1(u, v), \lambda_2(u, v), \lambda_3(u, v))$, where $\lambda_1(u, v)^2 + \lambda_2(u, v)^2$ is the square of a rational function.

3 Reparametrizing: some basic issues

The starting point for our approach, our input, is a rational parametrization of a true surface over the complexes of the form,

$$(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$$

with complex coefficients. We can imagine, from the context where the parametrization has arisen or from the way it has been obtained, that this parametrizes a swung surface over \mathbb{R}^3 . But, strictly speaking, our only mathematical data is the given parametrization. Since it has complex coefficients, all we can assert is that it parametrizes a surface $\mathcal{S}_{\mathbb{C}}$ in \mathbb{C}^3 .

In this Section we will deal with two basic issues that we have already mentioned in the Introduction: *a)* the precise meaning of the word “real” when applied to a complex surface, since it will be a basic requirement for our results and, *b)* the proper versus improper character of the given parametrization.

We recall that a parametrization is called *proper* or *birational* if the map from parameters to points in the surface is generically one-to-one, i.e. it is possible to invert the parametrization and to obtain the parameters in terms of rational functions on the surface. Otherwise, that is, in the many-to-one case, we say that the parametrization is *improper* or *unirational*. For (real or complex) curves it is well known (Lüroth’s theorem, see, for instance [13]) that the existence of an improper rational mapping implies –and it is algorithmically easy to find– the existence of a birational parametrization [2]. Castelnuovo theorem states that any complex unirational surface is also rational. But this is not true for real surfaces. In fact, Example 2.3 provides a real surface (although not properly parametrizable as swung surface over the complexes) that has a real unirational parametrization, but can not have a real birational parametrization.

We start with the following easy observation that will be used later:

Remark 3.1. Assume that a given plane curve parametrization $(p_1(t), p_2(t))$ is proper over \mathbb{C} . Then, for every scalars $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, the parametrization $(\lambda p_1(t), \mu p_2(t))$ is also proper. Indeed, as field extensions, we have $\mathbb{C}(\lambda p_1(t), \mu p_2(t)) = \mathbb{C}(p_1(t), p_2(t)) = \mathbb{C}(t)$. Obviously, the result works for curves in any dimension.

331 Now observe that, given a swung surface parametrization

$$332 (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$$

333 we may consider diverse candidates for our trajectory and profile curves, namely
 334 adjusting constants: $(\lambda \phi_1(t), \phi_2(t))$ and $((1/\lambda) \psi_1(s), (1/\lambda) \psi_2(s))$, for each non-
 335 zero, complex, value of λ . However, as a consequence of the previous observa-
 336 tion, if for choice of λ the curves are proper, so they are for any other choice.

337 Bearing this in mind we can state the following

338 **Lemma 3.2.** *Assume that the parametrization of the surface*

$$339 (\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$$

340 *is proper. Then the parametrizations of the curves $\phi(t) = (\phi_1(t), \phi_2(t))$ and*
 341 *$\psi(s) = (\psi_1(s), \psi_2(s))$ are also proper.*

342 *Proof.* Indeed, suppose that $t = T_1(x, y, z)$, $s = T_2(x, y, z)$ is the inverse of the
 343 parametrization of the surface, i.e.,

$$344 t = T_1(\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$$

$$345 s = T_2(\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$$

347 in $\mathbb{C}(t, s)$. Take any s_0 such that $\psi_1(s_0)$, $\psi_2(s_0)$ and $T_1(y\psi_1(s_0), y\psi_2(s_0), z)$ are
 348 well defined. We claim that

$$349 \tilde{T}_1(y, z) := T_1(y\psi_1(s_0), y\psi_2(s_0), z)$$

350 is the inverse of the parametrization $(\phi_1(t), \phi_2(t))$. Indeed, note that

$$351 \tilde{T}_1(\phi_1(t), \phi_2(t)) = T_1(\phi_1(t) \psi_1(s_0), \phi_1(t) \psi_2(s_0), \phi_2(t)) = t$$

352 by the equations above, which shows that $\mathbb{C}(\phi_1(t), \phi_2(t)) = \mathbb{C}(t)$, that is, that
 353 the curve $(\phi_1(t), \phi_2(t))$ is birational. A similar (symmetric) argument shows
 354 that for a fixed t_0 , the function

$$355 \tilde{T}_2(x, y) := T_2(x\phi_1(t_0), y\phi_1(t_0), \phi_2(t_0))$$

356 is the inverse of the parametrization $(\psi_1(s), \psi_2(s))$ so that this curve is birational
 357 too. \square

358 **Remark 3.3.** Notice that the converse is false, that is, if both parametrizations
 359 $(\phi_1(t), \phi_2(t))$ and $(\psi_1(s), \psi_2(s))$ are birational, then it is not true, in general,
 360 that the parametrization

$$361 (\phi_1(t) \psi_1(s), \phi_1(t) \psi_2(s), \phi_2(t))$$

362 of the swung surface is also birational. For instance, in the Example 2.3 above,
 363 both the ellipse $(4t/(t^2 + 1), (t^2 - 1)/(t^2 + 1))$ and the circle $((s^2 - 1)/(s^2 +$

1), $2s/(s^2 + 1)$) parametrizations are birational, but the parametrization of the ellipsoid of revolution

$$\left(\frac{s^2 - 1}{s^2 + 1}, \frac{4t}{t^2 + 1}, \frac{2s}{s^2 + 1}, \frac{4t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1} \right)$$

is not, since it is not injective; all points of the ellipsoid being covered twice by the parametrization mapping because we are rotating the whole ellipse around the z -axis rather than only half of it.

In the formulation of our main result (see Theorem 4.2) we have just required the strictly weaker assumption that the involved ϕ and ψ curves are given by a proper parametrization (over the complexes). We recall that, given any parametrization of a swung surface, it is algorithmically easy to obtain another one, describing the same surface, verifying this condition. See [2].

On the other technical issue –the notion of real surface– we can start by recalling that the concept of (algebraic) surface over \mathbb{C}^3 is simple and well established in algebraic geometry. It is just the solution set (over the complexes) of a non-constant polynomial in three variables, with complex coefficients: its implicit equation $F(x, y, z) = 0$. At every point, the surface is either locally diffeomorphic to an open ball of \mathbb{C}^2 (if we are at a regular point) or close, in the euclidean topology, to a regular point. This is the reason we say that a complex surface has (complex) dimension 2 (even considering that a ball in \mathbb{C}^2 is a 4-dimensional real object).

Given an algebraic surface $\mathcal{S}_{\mathbb{C}}$ in \mathbb{C}^3 , its real points $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ might yield a two dimensional subset of \mathbb{R}^3 , but it could also be just some geometric object of smaller (real) dimension or even empty. This is clearly the case if its implicit equation involves non-real coefficients (such as the complex plane $x + y + iz = 0$, describing just a real line in \mathbb{R}^3). Having a real implicit equation (i.e. being *real-defined*) is a necessary condition to avoid this phenomena and try to guarantee a two dimensional real part of a complex surface. But it is not sufficient. Think, for instance, of the surfaces defined by $x^2 + y^2 + z^2 + 1 = 0$ or by $x^2 + y^2 + z^2 = 0$. In the first case, the solution set over \mathbb{R}^3 is just empty. In the second case, just the origin of coordinates, while, over the complex affine space \mathbb{C}^3 , both cases yield true surfaces (according to our definition above), in fact rational. Therefore, neither the solution set of $x^2 + y^2 + z^2 + 1 = 0$ nor of $x^2 + y^2 + z^2 = 0$ are parametrizable with real coefficients, since if such parametrization would exist, it would yield –for real values of the parameters– many real points in the surface. Since we are interested in learning when there is a reparametrization with real coefficients of a given complex parametric surface, it is natural that we rule out –at least– such cases.

Thus, given a complex algebraic surface $\mathcal{S}_{\mathbb{C}}$ in \mathbb{C}^3 , we would like to name it as *real* if every (complex) polynomial vanishing over the set $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ must also vanishes over $\mathcal{S}_{\mathbb{C}}$. That is, if, in this sense, the real part of $\mathcal{S}_{\mathbb{C}}$ is algebraically indistinguishable from the whole complex surface. More technically, this condition is expressed by saying that the closure of $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$ in the Zariski topology is equal to $\mathcal{S}_{\mathbb{C}}$, $\mathcal{S}_{\mathbb{C}} = \overline{\mathcal{S} \cap \mathbb{R}^3}$. Clearly, none of the surfaces $\mathcal{S}_{\mathbb{C}} = x^2 + y^2 + z^2 + 1 = 0$

407 or $\mathcal{S}_{\mathbb{C}} = x^2 + y^2 + z^2 = 0$ are real, since, in the first instance, 1 is a polynomial
 408 vanishing over $\mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$, but not on $\mathcal{S}_{\mathbb{C}}$ and, in the second, $x = 0$ is an equa-
 409 tion holding over the real part, but not over the complex surface. For another
 410 example, let us consider a complex surface implicitly defined by a non-real poly-
 411 nomial, such as the plane $x + y + iz = 0$. It has many real points, but they
 412 verify simultaneously the two equations $\{x + y + iz = 0, \quad x + y - iz = 0\}$ and,
 413 thus, the real part of this surface verifies the system $\{x + y = 0, \quad z = 0\}$, which,
 414 obviously, does not apply to the whole complex plane. We conclude that this
 415 plane is not real.

416 Although here we are attempting to reduce technicalities to a minimum,
 417 the study of geometric objects defined as real solutions of polynomial equations
 418 belongs to the field of real algebraic geometry and we would like to point out at
 419 least some references for further details on this subject, such as the foundational
 420 book [6], or the paper [15], which addresses the so-called *complexification* of a
 421 real algebraic set. With this terminology, we will say that a (complex) surface $\mathcal{S}_{\mathbb{C}}$
 422 is *real* if it coincides with the complexification of its real part. For a *real* surface
 423 it is easy to prove that its real part is truly a surface, an object of real dimension
 424 two, in the sense of having points (in fact most of them) at which the surface is
 425 locally diffeomorphic to an open ball of \mathbb{R}^2 . However, contrary to what happens
 426 in \mathbb{C}^3 , this does not mean, in general, over \mathbb{R}^3 , that such points are dense, in
 427 the euclidean topology, over the real part of the surface. For example, consider
 428 the (absolutely) irreducible *real* surface $\mathcal{S}_{\mathbb{C}}$ given by $x^2(1 - x) + y^2 + z^2 = 0$.
 429 Then, it happens that \mathcal{S} is a 2-dimensional piece plus the origin, as an isolated
 430 (also in the Euclidean topology) real point.

431 Yet, with some simple algebraic considerations one can show that, for an irre-
 432 ducible complex surface, it is equivalent to be *real* and to have a two-dimensional
 433 real part (i.e. what one would expect to be “really” a real surface). From a
 434 computational point of view, there is an easy criterion to detect whether an ir-
 435 reducible complex surface (such as those given by a rational parametrization) is
 436 *real*. It is enough to detect the existence of a regular point which lies in \mathbb{R}^3 . (A
 437 point is regular if it is not a zero simultaneously of the equation of the surface
 438 and the derivatives of this equation with respect to the three variables x, y, z).
 439 See Proposition 1 in [16] or the basic reference on the topic, [6]. This is the test
 440 we have performed in Example 2.3 to conclude the reality of the offset surface.

441 If a surface $\mathcal{S}_{\mathbb{C}}$ is parametrizable with real rational functions, say, $f_1(t, s)$,
 442 $f_2(t, s)$, $f_3(t, s)$ in $\mathbb{R}(t, s)$, then it is real. In fact if a polynomial $G(x, y, z)$ van-
 443 ishes over $\mathcal{S} = \mathcal{S}_{\mathbb{C}} \cap \mathbb{R}^3$, it vanishes over all points $(f_1(t_0, s_0), f_2(t_0, s_0), f_3(t_0, s_0))$,
 444 with $t_0, s_0 \in \mathbb{R}$. Then $G(f_1(t, s), f_2(t, s), f_3(t, s))$ must be identically zero, hence,
 445 $G(x, y, z)$ vanishes over all $\mathcal{S}_{\mathbb{C}}$. As pointed out in the Introduction, it is unknown,
 446 in general, whether a complex parametrizable surface $\mathcal{S}_{\mathbb{C}}$ which is real, is also
 447 parametrizable by real rational functions. Our main result shows, that this is
 448 true in the particular case of parametrized swung surfaces.

449 For curves the situation is completely understood. As above, a (complex)
 450 curve is called real if every polynomial vanishing over all its real points must also
 451 vanish over the complex points of the curve, or, equivalently (in the irreducible
 452 case) the curve has infinitely many real points, or, equivalently, the subset of

453 real points is one dimensional, or it contains a real regular point, etc. Contrary
 454 to the case of surfaces, it is well known that a complex parametrizable curve
 455 has a real parametrization if and only if it is real, and we know how to find such
 456 a parametrization, [12]. This is the basis for the proof of our main result.

457 4 Reparametrizing swung surfaces

458 This section is devoted to present the main reparametrization result for *swung*
 459 surfaces. The problem of reparametrizing $\mathcal{S}_{\mathbb{C}}$ with rational functions having
 460 only real coefficients will be reduced, in essence, to the case of reparametrizing
 461 the involved curves ϕ and ψ . Then, for these curves, we will apply the real
 462 version of Lüroth theorem, using hypercircles, as in [12, 13]:

463 **Theorem 4.1** ([12]). *Let \mathcal{C} be a rational curve (over the complexes) given by a*
 464 *proper parametrization $\phi(t)$ with complex coefficients. There are equivalent:*

- 465 1. \mathcal{C} is \mathbb{R} -parametrizable.
- 466 2. There exists a change of parameter $s \rightarrow t = \xi(s) = \frac{as+b}{cs+d}$, with $a, b, c, d \in$
 467 \mathbb{C} , and $ad - bc \neq 0$, such that $\phi(\xi(s))$ has real coefficients.
- 468 3. \mathcal{C} is a real curve.

469 Moreover, there is an algorithm that taking as input the given parametrization
 470 ϕ determines if these equivalent conditions hold and, if so, computes the change
 471 of variables $t = \xi(s)$.

472 However, some complications arise. Consider, for instance, the surface $\mathcal{S}_{\mathbb{C}} :=$
 473 $\{yz + x^2 = 0\}$, parametrized by $\mathcal{P}(s, t) = (\mathbf{i}ts, ts^2, t)$. Then we may think of \mathcal{P}
 474 as a swung surface as in (§) with $\phi(t) = (\mathbf{i}t, t)$, $\psi(s) = (s, -\mathbf{i}s^2)$, so that neither
 475 ϕ nor ψ describes a real curve. However, we may also consider \mathcal{P} as described
 476 by $\phi'(t) = (t, t)$, $\psi'(s) = (\mathbf{i}s, s^2)$ and, then, both curves are real (the latter is
 477 the parabola $y + x^2 = 0$) and, thus, $\mathcal{P}(s, t)$ will be reparametrizable over the
 478 reals. Luckily, this example shows the general way to proceed. Next statement
 479 is the main result in the article.

480 **Theorem 4.2.** *Let $\mathcal{S}_{\mathbb{C}}$ be a rational complex surface, other than a plane, pa-*
 481 *rametrized by $\mathcal{P}(s, t)$. Let $(\phi_1(t), \phi_2(t)) \in \mathbb{C}(t)^2$ and $(\psi_1(s), \psi_2(s)) \in \mathbb{C}(s)^2$ be*
 482 *any proper parametrization of curves such that*

$$483 \quad \mathcal{P}(s, t) = (\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t)) \in \mathbb{C}(s, t)^3.$$

484 Then, the following statements are equivalent:

- 485 1. There exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that the curves defined by the parametriza-
- 486 tions $\phi_{\lambda} = (\lambda\phi_1(t), \phi_2(t))$ and $\psi_{\lambda} = (\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are \mathbb{R} -parameteri-
- 487 zable.

488 2. There exists a change of variables:

$$489 \quad \begin{aligned} \xi : \quad \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (u, v) &\mapsto \left(\frac{a_1 u + b_1}{c_1 u + d_1}, \frac{a_2 v + b_2}{c_2 v + d_2} \right) \end{aligned}$$

490 where $a_i b_i - c_i d_i \neq 0$, $i = 1, 2$, such that $\mathcal{P}(\xi(u, v)) \in \mathbb{R}(u, v)^3$.

491 3. $\mathcal{S}_{\mathbb{C}}$ is \mathbb{R} -parametrizable.

492 4. $\mathcal{S}_{\mathbb{C}}$ is a real surface.

493 The proof 4.2 requires some technical results related to the the construction
494 of the *parametric variety of Weil* associated to the given parametrization of the
495 swung surface. The detailed proof of the technical results has been included in
496 an Appendix.

497 *Proof.* 1. \rightarrow 2. If there is a λ such that the curves $(\lambda\phi_1(t), \phi_2(t))$ and $(\frac{1}{\lambda}\psi_1(s),$
498 $\frac{1}{\lambda}\psi_2(s))$ are \mathbb{R} -parametrizable, then, using the real Lüroth theorem [13] there
499 exists a change of parameters $u \rightarrow s(u) = \frac{a_1 u + b_1}{c_1 u + d_1}$, $v \rightarrow t(v) = \frac{a_2 v + b_2}{c_2 v + d_2}$, with
500 $a_i b_i - c_i d_i \neq 0$, $i = 1, 2$, such that $(\lambda\phi_1(t(v)), \phi_2(t(v)))$, $(\frac{1}{\lambda}\psi_1(s(u)), \frac{1}{\lambda}\psi_2(s(u)))$
501 are real parametrizations, so we take $\xi(u, v) = (s(u), t(v))$ and

$$502 \quad \mathcal{P}(s(u), t(v)) = \left(\lambda\phi_1(t(v)) \frac{1}{\lambda}\psi_1(s(u)), \lambda\phi_1(t(v)) \frac{1}{\lambda}\psi_2(s(u)), \phi_2(t(v)) \right) \in \mathbb{R}(u, v)^3$$

503 is real.

504 It is clear that 2. \rightarrow 3. and 3. \rightarrow 4., so we are left with proving that if the surface
505 is real, then, for a suitable $\lambda \neq 0$, $(\lambda\phi_1(t), \phi_2(t))$ and $(\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ define
506 \mathbb{R} -parametrizable curves.

507 In this direction, we will consider the specific parametric variety of Weil V
508 (see the Appendix) associated to the parametrization $\mathcal{P}(t, s)$. By definition,
509 this variety is obtained as follows. First, in the parametrization of $\mathcal{S}_{\mathbb{C}}$, per-
510 form the substitution $s := s_0 + \mathbf{i}s_1$ and $t := t_0 + \mathbf{i}t_1$, where s_0, s_1, t_0, t_1 are
511 new variables. Then, after some normalization, we get $\mathcal{P}(s_0 + \mathbf{i}s_1, t_0 + \mathbf{i}t_1) =$
512 $(\mathcal{P}_1(\bar{s}, \bar{t}), \mathcal{P}_2(\bar{s}, \bar{t}), \mathcal{P}_3(\bar{s}, \bar{t}))$, where

$$513 \quad \begin{aligned} \mathcal{P}_1(\bar{s}, \bar{t}) &= \frac{[A_0(\bar{t}) + \mathbf{i}A_1(\bar{t})]}{A(\bar{t})} \frac{[C_0(\bar{s}) + \mathbf{i}C_1(\bar{s})]}{C(\bar{s})} \\ \mathcal{P}_2(\bar{s}, \bar{t}) &= \frac{[A_0(\bar{t}) + \mathbf{i}A_1(\bar{t})]}{A(\bar{t})} \frac{[D_0(\bar{s}) + \mathbf{i}D_1(\bar{s})]}{D(\bar{s})} \\ \mathcal{P}_3(\bar{s}, \bar{t}) &= \frac{[B_0(\bar{t}) + \mathbf{i}B_1(\bar{t})]}{B(\bar{t})} \end{aligned}$$

514 with $A_i(\bar{t})$, $B_i(\bar{t})$, $A(\bar{t})$, $B(\bar{t}) \in \mathbb{R}[\bar{t}]$, $C_i(\bar{s})$, $D_i(\bar{s})$, $C(\bar{t})$, $D(\bar{t}) \in \mathbb{R}[\bar{s}]$, $\bar{s} =$
515 (s_0, s_1) and $\bar{t} = (t_0, t_1)$. Notice that the A 's and B 's arise from the substi-
516 tution in ϕ_1 and ϕ_2 and likewise the C 's and D 's come from the substitution in
517 ψ_1 and ψ_2 .

518 Second, we take the Zariski closure V of the open set given by:

$$\begin{aligned}
A_0(\bar{t})C_1(\bar{s}) + A_1(\bar{t})C_0(\bar{s}) &= 0 \\
A_0(\bar{t})D_1(\bar{s}) + A_1(\bar{t})D_0(\bar{s}) &= 0 \\
B_1(\bar{t}) &= 0 \\
A(\bar{t}) \neq 0, B(\bar{t}) \neq 0, C(\bar{s}) \neq 0, D(\bar{s}) \neq 0
\end{aligned}$$

where the first three equations correspond to the vanishing of the imaginary parts of the numerators of \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 . Notice that V does not depend on the precise choice of ϕ and ψ (by adjusting constants), but only on their product.

We have naturally the map:

$$\begin{aligned}
\mathcal{P}^* : \quad V &\rightarrow \mathcal{S}_{\mathbb{C}} \\
(s_0, s_1, t_0, t_1) &\mapsto (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) = \mathcal{P}(s_0 + \mathbf{i}s_1, t_0 + \mathbf{i}s_1)
\end{aligned}$$

From the definition of V , it is clear that \mathcal{P}^* carries real points of V to real points of $\mathcal{S}_{\mathbb{C}}$. Now, since $\mathcal{S}_{\mathbb{C}}$ is real, Theorem A.2 assures the existence of a real 2-dimensional component U of V such that $\mathcal{P}^* : U \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant.

Then, consider the matrix

$$M = \begin{pmatrix} A_0(t_0, t_1) & A_1(t_0, t_1) \\ -C_0(s_0, s_1) & C_1(s_0, s_1) \\ -D_0(s_0, s_1) & D_1(s_0, s_1) \end{pmatrix}$$

Notice that no row of M can be identically zero in U , since \mathcal{P}^* is dominant and $\mathcal{S}_{\mathbb{C}}$ is not a plane. For any point $p = (a_0, a_1, b_0, b_1)$ in the nonempty open subset of U such that $(A_0, A_1)(p) \neq (0, 0)$ we have that $\text{rank}(M) = 1$. Thus, if $A_1 \equiv 0$ in U , it follows that, $M \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in U . If $A_1 \not\equiv 0$ in U , then $(A_0/A_1)(t_0, t_1) = (-C_0/C_1)(s_0, s_1) = (-D_0/D_1)(s_0, s_1)$ is a real rational function in U . By Theorem A.4, U is a Cartesian product of two irreducible curves, so, by Lemma A.5, $(A_0/A_1)(t_0, t_1) = (-C_0/C_1)(s_0, s_1) = (-D_0/D_1)(s_0, s_1) = r \in \mathbb{R}$ and $M \cdot \begin{pmatrix} 1 \\ -r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in U .

In any case, there is a vector $(r_1, r_0) \in \mathbb{R}^2$, $(r_1, r_0) \neq (0, 0)$, such that $M \cdot \begin{pmatrix} r_1 \\ r_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in U .

Let $\lambda = r_0 + \mathbf{i}r_1$. We are going to prove that the curves defined by $\phi_\lambda = (\lambda\phi_1(t), \phi_2(t))$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1(s), \frac{1}{\lambda}\psi_2(s))$ are \mathbb{R} -parametrizable. Indeed,

$$\begin{aligned}
(r_0 + \mathbf{i}r_1) \frac{A_0(\bar{t}) + \mathbf{i}A_1(\bar{t})}{A(\bar{t})} &= \frac{r_0A_0(\bar{t}) - r_1A_1(\bar{t}) + \mathbf{i}(r_1A_0(\bar{t}) + r_0A_1(\bar{t}))}{A(\bar{t})} \\
\left(\frac{r_0 - \mathbf{i}r_1}{r_0^2 + r_1^2} \right) \frac{C_0(\bar{s}) + \mathbf{i}C_1(\bar{s})}{C(\bar{s})} &= \frac{r_0C_0(\bar{s}) + r_1C_1(\bar{s}) + \mathbf{i}(r_0C_1(\bar{s}) - r_1C_0(\bar{s}))}{(r_0^2 + r_1^2)C(\bar{s})} \\
\left(\frac{r_0 - \mathbf{i}r_1}{r_0^2 + r_1^2} \right) \frac{D_0(\bar{s}) + \mathbf{i}D_1(\bar{s})}{C(\bar{s})} &= \frac{r_0D_0(\bar{s}) + r_1D_1(\bar{s}) + \mathbf{i}(r_0D_1(\bar{s}) - r_1D_0(\bar{s}))}{(r_0^2 + r_1^2)D(\bar{s})}
\end{aligned}$$

547 If $p = (a_0, a_1, b_0, b_1) \in U$ then, $(r_1 A_0 + r_0 A_1)(a_0, a_1) = 0$, $B_1(a_0, a_1) = 0$,
 548 $A(a_0, a_1) \neq 0$, $B(a_0, a_1) \neq 0$ and $(r_0 C_1 - r_1 C_0)(b_0, b_1) = 0$, $(r_0 D_1 - r_1 D_0)(b_0, b_1)$
 549 $= 0$, $C(b_0, b_1) \neq 0$, $D(b_0, b_1) \neq 0$. Hence $U = U_1 \times U_2$, where U_1 is contained
 550 in the parametric variety of Weil of ϕ_λ and U_2 is contained in the parametric
 551 variety of Weil of ψ_λ .

552 Since U is real, U_1, U_2 are real curves. It follows from the theory of hyper-
 553 circles [11, 13, 3] that ϕ_λ and ψ_λ are real curves and, hence, real parametrizable
 554 curves. \square

555 **Remark 4.3.** If the given swung parametrization \mathcal{P} is proper (see Section 3)
 556 and if some of the equivalent conditions of Theorem 4.2 hold, then the real
 557 parametrization $\mathcal{P}(\xi(s, t))$ described in item 2. is also proper.

558 **Remark 4.4.** In the hypotheses of the theorem we have explicitly discarded
 559 the case of planes. We can easily check whether the given parametrization
 560 $\mathcal{P}(s, t)$ of a surface corresponds to a plane by considering four generic points
 561 $\mathcal{P}(s_i, t_i), i = 1 \dots 4$, and verifying, by computing a determinant, if they are
 562 coplanar. On the other hand, if $\mathcal{S}_\mathbb{C}$ is a plane, it is clear that we can parametrize
 563 it over the reals if and only if it is real. However, items 1 and 2 in the statement
 564 above need not hold, see Example 5.4.

565 **Corollary 4.5.** Let $\mathcal{S}_\mathbb{C}$ be a rational revolution surface, parametrized by

$$566 \quad \mathcal{P}(s, t) = (\phi_1(t) \frac{s^2 - 1}{s^2 + 1}, \phi_1(t) \frac{2s}{s^2 + 1}, \phi_2(t)) \in \mathbb{C}(s, t)^3,$$

567 where $(\phi_1(t), \phi_2(t))$ is a proper parametrization of a curve. The following state-
 568 ments are equivalent:

- 569 1. The curve defined by $\phi(t) = (\phi_1(t), \phi_2(t))$ is \mathbb{R} -parametrizable (equiva-
 570 lently, it is real).
- 571 2. There exists a change of parameters with complex coefficients $\xi : \mathbb{C} \rightarrow \mathbb{C}$,
 572 where $\xi(t) = \frac{at + b}{ct + d}$ and $ad - bc \neq 0$, such that $\mathcal{P}(s, \xi(t)) \in \mathbb{R}(s, t)^3$.
- 573 3. $\mathcal{S}_\mathbb{C}$ is \mathbb{R} -parametrizable (but, perhaps, not necessarily with a proper param-
 574 etrization)
- 575 4. $\mathcal{S}_\mathbb{C}$ is a real surface.

576 *Proof.* The only nontrivial implication is 4. \rightarrow 1. Notice that, in this case, the
 577 parametrization determines uniquely the curve ϕ and $\psi = (\frac{s^2-1}{s^2+1}, \frac{2s}{s^2+1})$. Assume
 578 first that $\mathcal{S}_\mathbb{C}$ is not a plane. By Theorem 4.2, from 4. it follows that there is a
 579 $\lambda \in \mathbb{C}^*$ with ϕ_λ and ψ_λ , \mathbb{R} -parametrizable. Now, observe that for any $\lambda \in \mathbb{C}^*$,
 580 ψ_λ parametrizes the circle $x^2 + y^2 = 1/\lambda^2$, that is real if and only if λ is real.
 581 And ϕ_λ , with $\lambda \in \mathbb{R}^*$, is \mathbb{R} -parametrizable if and only if ϕ is \mathbb{R} -parametrizable.

582 On the other hand, suppose that $\mathcal{S}_\mathbb{C}$ is a real plane defined by the real
 583 equation $ax + by + cz = d$. Then $\phi_1(t)(a \frac{s^2-1}{s^2+1} + b \frac{2s}{s^2+1}) = d - c\phi_2(t)$. Now

584 $(a \frac{s^2-1}{s^2+1} + b \frac{2s}{s^2+1})$ must be a constant. Otherwise, since the second term of the
 585 equality above does not involve the s variable, it will imply that $d - c\phi_2(t)$
 586 is zero and, then, $\phi_1(t)$ must be zero (but then $\mathcal{S}_{\mathbb{C}}$ is not a surface). Now, if
 587 $(a \frac{s^2-1}{s^2+1} + b \frac{2s}{s^2+1})$ is a constant, it must be $a = b = 0$. Thus $d - c\phi_2(t)$ is zero.
 588 But c can not be zero (since then $a = b = c = 0$, and do not have a plane).
 589 Therefore $\phi_2(t) = d/c \in \mathbb{R}$ and, since the parametrization ϕ is proper, it can be
 590 reparametrized to $(t, d/c)$. \square

591 5 The algorithm and examples

592 In this section we present how to derive an algorithm to check whether a swung
 593 parametrization defines a real surface \mathcal{S} and, if it is the case, to compute a real
 594 parametrization of \mathcal{S} .

595 Since we already have algorithms to reparametrize real curves ([12]) given by
 596 complex parametrizations, we base the algorithm on the characterization (1) of
 597 Theorem 4.2. Given two curve parametrizations $\phi(t) = (\phi_1(t), \phi_2(t))$ and $\psi(s) =$
 598 $(\psi_1(s), \psi_2(s))$, the only problem left is computing, if it exists, a $\lambda \in \mathbb{C}^*$ such
 599 that $\phi_\lambda = (\lambda\psi_1, \psi_2)$ and $\psi_\lambda = (\frac{1}{\lambda}\psi_1, \frac{1}{\lambda}\psi_2)$ are real curves. One possible naive
 600 approach could be implicitizing one of the curves, by considering λ a parameter,
 601 and then adjusting the possible values of λ that make such implicit equation real.
 602 But that procedure would not guarantee (unless we use some Cylindric Algebraic
 603 Decomposition techniques, see [6]) that the curve is real (only that it is real-
 604 defined) and, anyway, we would like to avoid the implicitization computation,
 605 preferring to work directly with the given parametric input.

606 Our approach relies on the following key observation. Let $\phi(t) = (\phi_1(t),$
 607 $\phi_2(t))$ be a complex parametrization and t_0, t_1 new variables. Write $\phi_2(t_0 +$
 608 $it_1) = \frac{B_0(t_0, t_1) + iB_1(t_0, t_1)}{B(t_0, t_1)}$. If there is a λ such that ϕ_λ parametrizes a real
 609 curve, then the corresponding hypercircle Z_1 of ϕ_λ is a real circle or line and its
 610 implicit equation is a factor of $B_1(t_0, t_1)$ in $\mathbb{R}[t_0, t_1]$. This provides an algorithm
 611 to reparametrize \mathcal{P} over the reals.

612 Algorithm 5.1.

- 613 • **Input:** A complex swung parametrization \mathcal{P} of a surface $\mathcal{S}_{\mathbb{C}}$, different
 614 from a plane, such that there exists $\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{C}(t)^2$ and
 615 $\mu(s) = (\mu_1(s), \mu_2(s)) \in \mathbb{C}(s)^2$ parametrizations of curves such that

$$616 \quad \mathcal{P}(t, s) = (\eta_1(t)\mu_1(s), \eta_1(t)\mu_2(s), \eta_2(t)) \in \mathbb{C}(t, s)^3.$$

- 617 **Output:** A real parametrization $\mathcal{P}'(t, s)$ of $\mathcal{S}_{\mathbb{C}}$ or “The surface is not
 618 real”

- 619 1. **Compute** a pair $\eta(t), \mu(s)$ from \mathcal{P} , verifying the input structure.
- 620 2. **Reparametrize** $\eta(t)$ and $\mu(s)$ to proper parametrizations $\phi(t)$ and $\psi(s)$
 621 of the same curves.

622 3. **Write** $\phi_2(t_0 + \mathbf{i}t_1) = \frac{B_0(t_0, t_1) + \mathbf{i}B_1(t_0, t_1)}{B(t_0, t_1)}$

623 4. **Compute** the factors of degree 1 and/or of degree 2 (that correspond to
624 circles) of $B_1(t_0, t_1)$ in $\mathbb{R}[t_0, t_1]$.

625 5. **For each** factor f from step 4. **do**

626 (a) **Compute** a real parametrization $(v_0(t), v_1(t))$ of the line or circle
627 defined by f .

628 (b) Let $v(t) = v_0(t) + \mathbf{i}v_1(t)$

629 (c) **If** there exists a $\lambda_f \in \mathbb{C}^*$ such that $(\lambda_f \phi_1(v(t)), \phi_2(v(t)))$ is real
630 **then:**

631 i. Apply the real reparametrization algorithm for curves to $\psi_{\lambda_f} =$
632 $(1/\lambda_f \psi_1, 1/\lambda_f \psi_2)$.

633 ii. **If** ψ_{λ_f} is real and $u(s)$ is an invertible linear fraction such that
634 $\psi_{\lambda_f}(u(s))$ is real **then return** $(u(s), v(t))$.

635 6. **If** no factor f works **then return** “*The surface is not real*”.

636 We remark that the computations in steps 1 and 5 (c) are straightforward.
637 For instance, λ_f can be taken as the inverse of the leading coefficient of the
638 numerator of $\phi_1(v(t))$ when this fraction is written with monic denominator.
639 Step 2 can be carried out by standard techniques ([2]).

640 The main difficulty in this approach is step 4, in which we have to factor a
641 bivariate polynomial in $\mathbb{R}[t_0, t_1]$. We present an alternative that needs only to
642 manipulate the complex roots of a univariate polynomial.

643 If $\phi_\lambda = (\lambda\phi_1, \phi_2)$, $\lambda \in \mathbb{C}^*$ parametrizes a real curve \mathcal{C}_λ , then the complex
644 conjugate parametrization $\overline{\phi}_\lambda = (\overline{\lambda}\phi_1, \overline{\phi}_2)$ is also a proper parametrization of
645 \mathcal{C}_λ . Hence, there is a linear fraction $v' \in \mathbb{C}(t)$ such that $\overline{\phi}_2(v'(t)) = \phi_2(t)$,
646 $\overline{\lambda}\phi_1(v'(t)) = \lambda\phi_1(t)$. For all but finitely many values t_0 of t , we have that
647 $\lambda/\overline{\lambda} = \phi_1(v'(t_0))/\phi_1(t_0)$.

648 The idea to compute the possible values of $\lambda/\overline{\lambda}$ is the following. First,
649 we choose a $t_0 \in \mathbb{C}$. Compute $\phi_2(t_0) = a_0$. The possible values of $u'(t_0)$
650 are the solutions b_j in \mathbb{C} of the univariate equation $\overline{\phi}_2(x) = a_0$. This will
651 give a set $A_{t_0} = \{b_1, \dots, b_d\}$. Now, the possible values of $\lambda/\overline{\lambda}$ are $S_{t_0} =$
652 $\{\overline{\phi}_1(b_1)/\phi_1(a_0), \dots, \overline{\phi}_1(b_d)/\phi_1(a_0)\}$. Note that $\lambda/\overline{\lambda}$ always has norm 1 so we
653 can take in S_{t_0} only those values of norm 1. On the other hand, from $\lambda/\overline{\lambda}$ we
654 can recover λ up to a real constant and thus, we get a finite set of candidates to
655 a λ verifying item 1. in Theorem 4.2. This description alone already provides
656 an algorithm. For every candidate λ , we apply the reparametrization algorithm
657 for ϕ_λ and ψ_λ .

658 In practice, except for rare cases, S_{t_0} is either empty (and $\mathcal{S}_\mathbb{C}$ is not real) or
659 it is already the complete set of valid $\lambda/\overline{\lambda}$. Moreover, it is, typically, a singleton.
660 If $r \in S_{t_0}$, $r = r_0 + \mathbf{i}r_1 \in \mathbb{C}$ then $r_0^2 + r_1^2 = 1$ and $\lambda/\overline{\lambda} = r_0 + \mathbf{i}r_1$. If $r = 1$ a
661 solution is $\lambda = 1$. If $r \neq 1$ a solution is $\lambda = r_1 + \mathbf{i}(1 - r_0) \in \mathbb{C}^*$.

662 There are only two possible kinds of t_0 values where this procedure to com-
663 pute λ does not work. First, when $\phi(t_0)$ is not defined (because the denominator

vanishes). The other case is if $\phi_2(t_0) = \overline{\phi_2}(\infty)$. But these are $2d$ cases that can be discarded easily.

Once the possible λ 's are computed, we only have to check, for each λ , if ϕ_λ and ψ_λ are real and, if so, to compute a real reparametrization.

We can use this discussion to derive an algorithm that either checks that ϕ_λ is never a real curve or returns the values λ such that ϕ_λ is real. We must point out that this approach will not work if ϕ is a horizontal or vertical line. But these are corner cases that can be easily solved by direct means.

The description of this alternative algorithm (without emphasizing corner cases) could be:

Algorithm 5.2.

• **Input:** A complex swung parametrization \mathcal{P} of a surface $\mathcal{S}_{\mathbb{C}}$, different from a plane, such that there exists $\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{C}(t)^2$ and $\mu(s) = (\mu_1(s), \mu_2(s)) \in \mathbb{C}(s)^2$ parametrizations of curves such that

$$\mathcal{P}(s, t) = (\eta_1(t)\mu_1(s), \eta_1(t)\mu_2(s), \eta_2(t)) \in \mathbb{C}(t, s)^3.$$

Output: A real parametrization $\mathcal{P}'(t, s)$ of $\mathcal{S}_{\mathbb{C}}$ or “The surface is not real”

1. **Compute** a pair $\eta(t)$, $\mu(s)$ from \mathcal{P} , verifying the input structure.
2. **Reparametrize** $\eta(t)$ and $\mu(s)$ to proper parametrizations $\phi(t)$ and $\psi(s)$ of the same curves.
3. **Compute** the complex conjugates $\overline{\phi_1}$, $\overline{\phi_2}$ of ϕ .
4. **Compute** $a_\infty = \overline{\phi_2}(\infty) \in \mathbb{C} \cup \{\infty\}$
5. $S \leftarrow \mathbb{C}$
6. **while** $S = \mathbb{C}$ **do**
 - (a) $a \leftarrow \text{random}(\mathbb{C})$
 - (b) $b \leftarrow \phi_2(a)$
 - (c) **If** $b \neq \infty$ **and** $b \neq a_\infty$ **then**
 - i. $T \leftarrow \{t \in \mathbb{C} \mid \overline{\phi_2(t)} = b\}$
 - ii. $S \leftarrow S \cap \{s = \overline{\phi_1(t)}/\phi_1(a) \mid t \in T, |s| = 1\}$
7. **If** $S = \emptyset$ **then return** “ ϕ_λ is never real”
8. $\Lambda \leftarrow \emptyset$
9. **For each** $r = r_0 + ir_1 \in S$ **do**
 - (a) **If** $r = 1$ **then** $\lambda \leftarrow 1$ **else** $\lambda \leftarrow r_1 + i(1 - r_0)$.
 - (b) $\Lambda = \Lambda \cup \{\lambda\}$

698 10. **for each** λ in Λ **do**
699 (a) Compute (if possible) u, v such that $\phi_\lambda(v), \psi_\lambda(s)$ are real.
700 11. No pair (u, v) is found **then return** “ ϕ_λ is never real” **else return** “pairs
701 $(u(s), v(t))$ found”.

702 This alternative algorithm has some advantages over the first one. Along
703 the paper, including the algorithms, it is assumed that we are working in a
704 field $\mathbb{K}(\mathbf{i})$ where computations are exact (infinite precision). However, the case
705 that the input is given by a floating point approximation is also interesting. In
706 this context, if we apply Algorithm 5.1, we should be dealing with an approxi-
707 mate factorization of $B_1(t_0, t_1)$ over the reals. On the other hand, Algorithm 5.2
708 would have to compute all complex roots of some univariate polynomials, a more
709 common problem. We have made experiments with the math software Sage using
710 both algorithms for inputs in $\mathbb{Q}(\mathbf{i})$ and with floating point arithmetic. The
711 running times are described in Table 1. Case 1 is Algorithm 5.1 in $\mathbb{Q}(\mathbf{i})$ and ex-
712 act computations. Case 2 is Algorithm 5.2 also in $\mathbb{Q}(\mathbf{i})$ and exact computations.
713 Finally, Case 3 is Algorithm 5.2 using floating point arithmetic. The tests are
714 performed as following. First, we construct two random rational parametriza-
715 tions $\phi_r = (\phi_1(t), \phi_2(t))$ and $\psi_r = (\psi_1(s), \psi_2(s))$, of degree d and coefficients
716 over \mathbb{Q} . The tested degrees for ϕ and ψ have been $d = 1, 2, 5, 10, 25$. Then we
717 compute random linear fractions $u(s), v(t)$ with coefficients in $\mathbb{Q}(\mathbf{i})$. Finally, the
718 input is $\mathcal{P} = (\phi_1(v(t))\psi_1(u(s)), \phi_1(v(t))\psi_2(u(s)), \phi_2(v(t)))$. We have prepared
719 three tables considering a bound for the size of the integers in ϕ and ψ , with
720 bounds $2^8, 2^{16}$ and 2^{32} respectively. In all cases, the coefficients of u and v are
721 bounded by 100, so we know before hand that in all cases there are solutions
722 with *small height*. Note that these figures are not the bound of the input \mathcal{P} ,
723 since we have to perform a composition and a multiplication. For instance, the
724 bigger case is degree 25 and initial coefficients bounded by 2^{32} , yielding the final
725 size of the coefficients of the input \mathcal{P} around 2^{1700} .

726 By looking into the tables we observe that Algorithm 5.2 behaves similarly
727 to Algorithm 5.1 for reasonable degrees. But, for very big degrees or very big
728 coefficients, Algorithm 5.2 performs better.

729 On the other hand, we notice that using floating point arithmetic is much
730 faster. What we get as output in this case is a couple of linear fractions
731 $(u(s), v(t))$ such that, for (s, t) real parameters, $\mathcal{P}(u(s), v(t))$ has a very small
732 imaginary part (i.e. as if it were real, in practice). In the floating point case,
733 as the degree grows, the numerical error increases to the point that, for degree
734 25, our implementation sometimes fail. Each case has been executed ten times
735 and we display, in the corresponding entry of the table, both the best and worst
736 obtained time in seconds.

size 2^8	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.25-0.42	0.49-0.52	0.64-0.78	1.22-1.36	5.45-14.15
Case 2	0.52-0.55	0.65-0.69	0.77-1.01	1.11-1.19	2.88-3.12
Case 3	<0.01	<0.01	0.013-0.015	0.02-0.03	0.15-0.19
size 2^{16}	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.72-0.78	0.88-0.91	1.06-1.34	2.01-2.18	41.78-52.7
Case 2	0.39-0.41	0.51-0.66	0.64-0.68	1.06-1.11	3.71-3.94
Case 3	<0.01	0.01	0.013-0.015	0.02-0.03	0.14-0.15
size 2^{32}	deg. 1	deg. 3	deg. 5	deg. 10	deg. 25
Case 1	0.96-0.99	0.72 - 1.16	1.43-1.47	3.31-3.59	>60
Case 2	0.53-0.56	0.75-0.79	0.91-0.97	1.51-1.56	6.02-6.71
Case 3	<0.01	<0.01	0.013-0.015	0.03	0.14-0.16

Table 1: Running time of the algorithms

5.1 Examples

Example 5.3. Let $\mathcal{S}_{\mathbb{C}}$ be the classical revolution surface given by the parametrization

$$\left(\frac{3-t^2}{4-2t} \frac{s^2-1}{s^2+1}, \frac{3-t^2}{4-2t} \frac{2s}{s^2+1}, \frac{-it^2+4it-3i}{2t-4} \right)$$

If we take the ϕ -curve parametrized by $(\frac{3-t^2}{4-2t}, \frac{-it^2+4it-3i}{2t-4})$ and perform the method in [12], we obtain that we have to parametrize the circle $x^2 + y^2 - 4x + 3 = 0$ (and, thus, the given curve is real), yielding the associated unit $\xi(t) = (t + 3i)/(t + i)$. If we apply this unit to the original parametrization we get the following real parametrization of $\mathcal{S}_{\mathbb{C}}$:

$$\left(\frac{t^2+3}{t^2+1} \frac{s^2-1}{s^2+1}, \frac{t^2+3}{t^2+1} \frac{2s}{s^2+1}, \frac{2t}{t^2+1} \right)$$

Example 5.4. We now show that Theorem 4.2 does not work for planes. Consider the plane given by the parametrization

$$\mathcal{P} = ((it + 1)s, (it + 1)s, t)$$

Of course, this is the plane $\{x = y\}$, but if one computes the parametric variety of Weil as in the proof of 4.2, one gets $V = U = \{t_1 = 0, t_0 s_0 + s_1 = 0\}$, so U does not have the shape announced in Theorem A.4. This happens because $\mathcal{S}_{\mathbb{C}}$ is a plane, so items (1) and (2) of Theorem 4.2 do not apply. There is no $\lambda \in \mathbb{C}^*$ such that $(\lambda(it + 1), t)$, $(1/\lambda s, 1/\lambda s)$ are real curves. Still, U is \mathbb{R} -parametrizable by $t_0 = v, t_1 = 0, s_0 = u, s_1 = -uv$, so $\mathcal{P}(u, v - iuv) = (u^2 v + v, u^2 v + v, u) \in \mathbb{R}(u, v)^3$.

757 **Example 5.5.** Consider now the surface $xz - y^4$ given by the parametrization

$$758 \quad \mathcal{P}(s, t) = (\mathbf{i}ts^4, \mathbf{i}ts, -\mathbf{i}t^3)$$

759 The parametrization is not proper, but $(t, -\mathbf{i}t^3)$, $(\mathbf{i}s^4, \mathbf{i}s)$ are both proper. If
 760 we perform our method we get in V three valid components in the sense of
 761 Theorem A.1:

$$762 \quad U_1 = \{t_0 = 0, s_1 = 0\}, \lambda_1 = \mathbf{i}$$

$$764 \quad U_2 = \{t_0 - \sqrt{3}t_1 = 0, s_0 - \sqrt{3}/3s_1 = 0\}, \lambda_2 = \frac{\sqrt{3} - \mathbf{i}}{2}$$

$$765 \quad U_3 = \{t_0 + \sqrt{3}t_1 = 0, s_0 + \sqrt{3}/3s_1 = 0\}, \lambda_3 = \frac{-\sqrt{3} - \mathbf{i}}{2}$$

767 Each U_i is a plane, parametrizable as

$$768 \quad U_1 : (0, t, s, 0)$$

$$769 \quad U_2 : (\sqrt{3}t, t, \sqrt{3}/3s, s)$$

$$770 \quad U_3 : (-\sqrt{3}t, t, -\sqrt{3}/3s, s)$$

773 Thus, we get three different reparametrizations of the original surface:

$$774 \quad \mathcal{P}_1 = (-ts^4, -ts, -t^3)$$

$$775 \quad \mathcal{P}_2 = \left(\frac{32}{9}ts^4, -\frac{4}{\sqrt{3}}ts, 8t^3 \right)$$

$$776 \quad \mathcal{P}_3 = \left(\frac{32}{9}ts^4, \frac{4}{\sqrt{3}}ts, 8t^3 \right)$$

779 **Example 5.6.** Similarly, if we start with the parametrization

$$780 \quad (\mathbf{i}ts^8, \mathbf{i}ts, -\mathbf{i}t^7)$$

781 and perform the algorithm, we find that there are seven valid components U . If
 782 we take $\phi = (t, -\mathbf{i}t^7)$, $\psi = (\mathbf{i}s^8, \mathbf{i}s)$, one of the components of U is associated
 783 to the value $\lambda = \mathbf{i}$ and the change of variables is $(u(s) = s, v(t) = \mathbf{i}t)$.

784 However, for the rest of components, we have that the other six values of λ
 785 are the complex roots of $x^6 - 5\mathbf{i}x^5 - 11x^4 + 13\mathbf{i}x^3 + 9x^2 - 3\mathbf{i}x - 1$. Each of
 786 these λ 's corresponds to the change of variables

$$787 \quad u(s) = (G(\lambda) + I)s, v(t) = (F(\lambda) + I)t$$

788 where

$$789 \quad F(\lambda) = (2144\lambda^{11} + 6096\lambda^9 + 18187\lambda^7 - 5532\lambda^5 + 52746\lambda^3 - 29068\lambda)/2059,$$

$$790 \quad G = (564\lambda^{11} + 1788\lambda^9 + 5687\lambda^7 + 404\lambda^5 + 18462\lambda^3 - 10520\lambda)/14413$$

Example 5.7. This is an example of floating point computation. Let
 $\mathcal{P} = ((((-0.235869421766 + 0.00479979499514i) t^2 s^2 + (-1.06313828776 - 0.166407418$
 $395i) t^2 s + (-0.2298109337 - 0.194094699602i) ts^2 + (-0.549385710585 - 0.417008231$
 $694i) t^2 + (-0.877430457459 - 1.05483464219i) ts + (1.66137786935 + 0.43565373369$
 $3i) s^2 + (-0.174582398271 - 0.861933962222i) t + (7.11424400952 + 3.28051289442i)$
 $s + 3.0177826152 + 4.01338551785i) / (t^2 s^2 + (1.86773892267 + 0.815610295477i) t^2 s$
 $+ (0.246950616172 + 0.659953272957i) ts^2 + (-0.629990211803 + 0.819708831258i) t^2$
 $+ (-0.0770254061546 + 1.43403588007i) ts + (-0.637235543476 - 0.0986590151152i)$
 $s^2 + (-0.696545997048 - 0.213336501249i) t + (-1.10972231899 - 0.704005152506i)$
 $s + 0.482323820976 - 0.46019338875i), ((0.043748011838 + 0.000454800948882i) t^2 s^2$
 $+ (-0.0131269921629 + 0.0392333791882i) t^2 s + (0.0414912865933 + 0.037287262802i)$
 $ts^2 + (-0.142191088123 + 0.00237065960661i) t^2 + (-0.0456137529341 + 0.0264953515$
 $089i) ts + (-0.305468984051 - 0.0902226718149i) s^2 + (-0.138098376489 - 0.11750813$
 $9573i) t + (0.169985946752 - 0.248640737912i) s + 1.00050177551 + 0.266290220991i)/$
 $(t^2 s^2 + (1.86773892267 + 0.815610295477i) t^2 s + (0.246950616172 + 0.659953272957i)$
 $ts^2 + (-0.629990211803 + 0.819708831258i) t^2 + (-0.0770254061546 + 1.43403588007i)$
 $ts + (-0.637235543476 - 0.0986590151152i) s^2 + (-0.696545997048 - 0.21333650124$
 $9i) t + (-1.10972231899 - 0.704005152506i) s + 0.482323820976 - 0.46019338875i),$
 $((-2.01273043888 + 0.00917837700067i) t^2 + (-2.39821706934 - 1.65257355305i) t +$
 $3.18517172695 + 0.210190888739i)/(t^2 + (0.246950616172 + 0.659953272957i) t - 0.6$
 $37235543476 - 0.0986590151152i))$

This is an approximate parametrization of a real surface. If we perform Algorithm 5.2, we get, as λ ,

$$\lambda = -0.999993922197720 + 0.00348648356104579i, u = ((121.322126428429 - 103.745283053666i)t - 103.745283053666 + 88.1900509458403i)/(t - i), v = ((75.1892967277426 - 78.1929832049560i)s - 78.1929832049560 + 80.4349108110022i)/(s - i).$$

With this unit, we get, for instance:

$$\mathcal{P}(u, v)(0, 2) = (-0.247210104423103 + 3.75195846613607 \times 10^{-11}i, 0.0416569932380774 + 5.64823188220487 \times 10^{-12}i, -2.00183575113046 + 3.09979819590467 \times 10^{-12}i)$$

which is “practically” real.

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A The parametric variety of Weil

The *parametric Weil* construction and the theory of *hypercircles and ultra-quadrics*, are tools developed in [11], [3]. Here we will consider the specific parametric variety of Weil V associated to the parametrization $\mathcal{P}(t, s)$ defined in the proof of Theorem 4.2 and the map

$$\begin{array}{ccc} \mathcal{P}^* : & V & \rightarrow \mathcal{S}_{\mathbb{C}} \\ & (t_0, t_1, s_0, s_1) & \mapsto \mathcal{P}(t_0 + \mathbf{i}t_1, s_0 + \mathbf{i}s_1) \end{array}$$

Recall that, by construction, \mathcal{P}^* carries real points of V to real points of $\mathcal{S}_{\mathbb{C}}$.

The importance of this variety V is that it encodes the fact that $\mathcal{S}_{\mathbb{C}}$ is real-defined or real parametrizable.

Theorem A.1. *Let V be the parametric variety of Weil associated to \mathcal{P} . If $\mathcal{S}_{\mathbb{C}}$ is a real-defined surface then there is (at least) one surface U that is an irreducible component of V such that $\mathcal{P}^* : U \rightarrow \mathcal{S}_{\mathbb{C}}$ is a dominant map. Moreover, if $\tau(u, v)$ is a real parametrization of U , then $\mathcal{P}^*(\tau(u, v))$ is a real parametrization of $\mathcal{S}_{\mathbb{C}}$.*

Proof. This is a direct consequence of Theorem 10 in [3]. \square

Note that, in the theorem, the surface U needs not be real-defined. By [3], Corollary 13, if $\mathcal{S}_{\mathbb{C}}$ is real-defined we know that there exists a real-defined surface W such that $\mathcal{P}^* : W \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant, but W needs not to be irreducible.

902 In our particular case we want to explore with more detail the surfaces U_i ,
 903 those components of V such that the map $\mathcal{P}^* : U_i \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant. Specially,
 904 we would like to understand the projections of such components into the (t_0, t_1)
 905 and (s_0, s_1) planes.

906 **Theorem A.2.** *If $\mathcal{S}_{\mathbb{C}}$ is a real surface, then there is a real irreducible surface*
 907 *U , a component of V , such that the map $\mathcal{P}^* : U \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant and \mathcal{P}^**
 908 *takes real points of U to real points of $\mathcal{S}_{\mathbb{C}}$.*

909 *Proof.* By [3], $\mathcal{P}^* : V \rightarrow \mathcal{S}_{\mathbb{C}}$ is generically (over an nonempty open subset of
 910 $\mathcal{S}_{\mathbb{C}}$) finite to one. So, if U_i is a component of V of dimension different from 2,
 911 then $\mathcal{P}^* : U_i \rightarrow \mathcal{S}_{\mathbb{C}}$ is not dominant. Let U' be the union of all the components
 912 W of V such that the map $\mathcal{P}^* : W \rightarrow \mathcal{S}_{\mathbb{C}}$ is not dominant. In particular, U'
 913 contains all components of V that are not surfaces. Then $\mathcal{P}^*(U')$ is contained
 914 in a 1-dimensional subset of $\mathcal{S}_{\mathbb{C}}$. Let $\{U_1, \dots, U_k\}$ be the remaining components
 915 of V . Each U_i is a surface and $\mathcal{P}^* : U_i \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant. By Theorem A.1
 916 there is at least one such surface U_i .

917 Consider now the set $\mathcal{S}'_{\mathbb{C}} = \mathcal{P}^*(V) - \mathcal{P}^*(U') \subseteq \mathcal{S}_{\mathbb{C}}$. This is a subset of $\mathcal{S}_{\mathbb{C}}$
 918 that contains a non-empty open Zariski subset of $\mathcal{S}_{\mathbb{C}}$ (Shafarevich, Chapter 1,
 919 §5, Theorem 6). It follows that the set of real points of $\mathcal{S}'_{\mathbb{C}}$ is Zariski-dense in
 920 $\mathcal{S}_{\mathbb{C}}$.

921 Let $p = (p_1, p_2, p_3)$ be a real point of $\mathcal{S}'_{\mathbb{C}}$. Since $p \in \mathcal{P}^*(V)$, then $p = \mathcal{P}(a, b)$,
 922 for some $a = a_0 + \mathbf{i}a_1$, $b = b_0 + \mathbf{i}b_1$, $a_0, a_1, b_0, b_1 \in \mathbb{R}$. Now

$$923 \quad \frac{A_0(a_0, a_1) + \mathbf{i}A_1(a_0, a_1)}{A(a_0, a_1)} \cdot \frac{C_0(b_0, b_1) + \mathbf{i}C_1(b_0, b_1)}{C(b_0, b_1)} = \phi_1(a)\psi_1(a) = p_1 \in \mathbb{R},$$

924 so

$$925 \quad A(a_0, a_1) \neq 0, C(b_0, b_1) \neq 0$$

926 and

$$927 \quad A_0(a_0, a_1)C_1(b_0, b_1) + A_1(a_0, a_1)C_0(b_0, b_1) = 0.$$

928 Analogously,

$$929 \quad D(b_0, b_1) \neq 0, B(a_0, a_1) \neq 0$$

930 and

$$931 \quad A_0(a_0, a_1)D_1(b_0, b_1) + A_1(a_0, a_1)D_0(b_0, b_1) = 0.$$

932 Thus, $(a_0, a_1, b_0, b_1) \in V \cap \mathbb{R}^4$. Moreover, $(a_0, a_1, b_0, b_1) \notin U'$, by our choice of
 933 p ; and $(a_0, a_1, b_0, b_1) \in U_1 \cup \dots \cup U_k$. Therefore, we have proved that any real
 934 point of $\mathcal{S}'_{\mathbb{C}}$ comes from at least one real point in $(a_0, a_1, b_0, b_1) \in U_1 \cup \dots \cup U_k$.
 935 If no U_i were real, then the set of real points $U_{i, \mathbb{R}}$ of each U_i would be contained
 936 in a 1-dimensional subset R_i of U_i . Then, the set of real points of $\mathcal{S}'_{\mathbb{C}}$ would be
 937 contained in $\mathcal{P}^*(R_1) \cup \dots \cup \mathcal{P}^*(R_k)$, which is included in a dimension 1 subset
 938 of $\mathcal{S}'_{\mathbb{C}}$, contradicting the fact that this set is Zariski dense in $\mathcal{S}_{\mathbb{C}}$.

939 So, there is at least one component U_i that is real. The fact that any real
 940 point of U_i maps to a real point of $\mathcal{S}_{\mathbb{C}}$ follows from the definition of V and
 941 \mathcal{P}^* . \square

942 With this result and bearing in mind the special shape of non planar swung
 943 surfaces, we can analyze the structure of the surfaces U_i in this case: they turn
 944 to be either planes, cylinders or tori. First, we need the following technical
 945 lemma:

946 **Lemma A.3.** *Consider the polynomial $f = C_0D_1 - C_1D_0 \in \mathbb{R}[s_0, s_1]$. If f is*
 947 *identically zero, then $\mathcal{S}_{\mathbb{C}}$ is a real plane.*

948 *Proof.* Since $\psi(t) = (\psi_1, \psi_2)$ is a proper parametrization of a curve, both com-
 949 ponents cannot be constants. Assume, without loss of generality, that ψ_2 is not
 950 constant, so D_0 and D_1 are not zero. Now, suppose that $C_0D_1 - C_1D_0 = 0$.
 951 Then $C_0/D_0 = C_1/D_1 = k(s_0, s_1)$. But, then, $C_0 + \mathbf{i}C_1 = k \cdot (D_0 + \mathbf{i}D_1)$ and

$$952 \quad \psi_1(s_0 + \mathbf{i}s_1) = \frac{C_0 + \mathbf{i}C_1}{C} = \frac{D_0 + \mathbf{i}D_1}{D} \cdot \frac{k \cdot D}{C} = \psi_2(s_0 + \mathbf{i}s_1) \cdot \frac{k \cdot D}{C}$$

953 So, $\frac{k \cdot D}{C} = \psi_1(s_0 + \mathbf{i}s_1)/\psi_2(s_0 + \mathbf{i}s_1)$ is both an \mathbf{i} -analytic rational function (i.e.,
 954 the expansion in terms of real and imaginary parts of the complex function
 955 $\psi_1(s)/\psi_2(s)$, after decomposing the variable s in real and imaginary terms,
 956 cf. [14]) and a real rational function. By the well known Cauchy-Riemann
 957 conditions for analyticity (cf. [14]), kD/C must be, then, a real constant r .
 958 Thus, $\psi_1 = r\psi_2$ and $\mathcal{S}_{\mathbb{C}}$ is the real plane $\{ry - x = 0\}$ in \mathbb{C}^3 . \square

959 **Theorem A.4.** *Let $\mathcal{S}_{\mathbb{C}}$ be a real swung surface, different from a plane, given*
 960 *by the parametrization \mathcal{P} . Let U be any irreducible surface in V such that*
 961 *$\mathcal{P}^* : U \rightarrow \mathcal{S}_{\mathbb{C}}$ is dominant. Then, there are irreducible curves $Z_1, Z_2 \subseteq \mathbb{C}^2$ such*
 962 *that $U = Z_1 \times Z_2$. Moreover, U is real if and only if both Z_1, Z_2 are real.*

963 *Proof.* Consider the two projections $\pi_1 : \mathbb{C}^4 \rightarrow \mathbb{C}^2$, $\pi_2 : \mathbb{C}^4 \rightarrow \mathbb{C}^2$, so that
 964 $\pi_1(t_0, t_1, s_0, s_1) = (t_0, t_1)$ and $\pi_2(t_0, t_1, s_0, s_1) = (s_0, s_1)$. Let Z_i be the Zariski
 965 closure of $\pi_i(U)$, $i = 1, 2$. Clearly, Z_1, Z_2 are irreducible varieties of \mathbb{C}^2 . If
 966 $\dim(Z_i)$ were 0, then $\mathcal{P}^*(U)$ would not be dense in $\mathcal{S}_{\mathbb{C}}$, contradicting the hy-
 967 pothesis. If $U = Z_1 \times Z_2$, then it is clear that U is real if and only if Z_1 and Z_2
 968 are real. Since always $U \subseteq Z_1 \times Z_2$ and both varieties are irreducible, to prove
 969 the theorem, it suffices to show that they have the same dimension, i.e. that
 970 $\dim(Z_i) \leq 1$, $i = 1, 2$.

971 Since $\mathcal{S}_{\mathbb{C}}$ is not a plane, $\phi_2(t)$ is not a constant, so, by [14], $B_1(t_0, t_1)$ is not
 972 a constant and $Z_1 \subseteq \{B_1(t_0, t_1) = 0\}$ has dimension at most 1.

973 Now, since $\psi = (\psi_1, \psi_2)$ is a curve, one of the components is not a constant.
 974 Assume, without loss of generality, that ψ_1 is not constant. Then, neither C_0
 975 nor C_1 are constants.

976 Now, we distinguish three cases. First, if $A_0 \equiv 0$ in U , then $A_1 \not\equiv 0$ in U ,
 977 because $\mathcal{P}^*(U)$ is dense in $\mathcal{S}_{\mathbb{C}}$. Since $A_0C_1 + A_1C_0 \equiv 0$ in U , it must happen
 978 that $C_0 \equiv 0$ in U , yielding $Z_2 \subseteq \{C_0 = 0\}$ and, thus, $\dim(Z_2) \leq 1$.

979 Analogously, if $A_1 \equiv 0$ in U , then $A_0 \not\equiv 0$ in U and $C_1 \equiv 0$ in U . Hence
 980 $Z_2 \subseteq \{C_1 = 0\}$ and $\dim(Z_2) \leq 1$.

981 Finally, assume that neither A_0 nor A_1 are zero in U , then

$$982 \quad A_0A_1(C_0D_1 - C_1D_0) = A_0D_1(A_1C_0 + A_0C_1) - A_0C_1(A_1D_0 + A_0D_1)$$

983 is zero in U . It follows that $C_0D_1 - C_1D_0 \equiv 0$ in U and $Z_2 \subseteq \{C_0D_1 - C_1D_0 =$
984 $0\}$. Since $\mathcal{S}_{\mathbb{C}}$ is not a plane, $C_0D_1 - C_1D_0$ is not identically zero (in \mathbb{C}^2) by
985 Lemma A.3 and, thus, $\dim(Z_2) \leq 1$. \square

986 Finally, we show another technical result:

987 **Lemma A.5.** *Let $U \subseteq \mathbb{C}^{n+m}$ be a real irreducible variety such that $U = U_1 \times U_2$*
988 *is the Cartesian product of two irreducible varieties $U_1 \subseteq \mathbb{C}^n$, $U_2 \subseteq \mathbb{C}^m$. Let*
989 *$F(\bar{x}, \bar{y}) \in \mathbb{R}(U)$ be a real rational function (i.e. $F(p) \in \mathbb{R}$, for any real point*
990 *where F is defined) such that it has two different representations $F(\bar{x}, \bar{y}) =$*
991 *$G(\bar{x}) = H(\bar{y})$. Then F is a real constant function equal to some $c \in \mathbb{R}$.*

992 *Proof.* Let $p_{x_0} \in U_1$ be a point such that $G(p_{x_0}) = c$ is defined. The fiber
993 $\{p_{x_0}\} \times U_2 \subseteq U$ is isomorphic to U_2 and, for any $p = (p_{x_0}, p_y) \in \{p_{x_0}\} \times U_2$,
994 we have that $F(p) = H(p_y) = G(p_{x_0}) = c$. Hence H is constant in U_2 and
995 $c = H(\bar{y}) = F(\bar{x}, \bar{y})$ is constant in U . Since both F and U are real, $c \in \mathbb{R}$. \square